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Abstract. A relation between the total instanton number and the quantum numbers of magnetic monopoles that arise in general Abelian gauges in $SU(2)$ Yang–Mills theory is established. The instanton number is expressed as the sum of the 'twists' of all monopoles, where the twist is related to a generalized Hopf invariant. The origin of a stronger relation between instantons and monopoles in the Polyakov gauge is discussed.

1. Introduction

Instantons and (Abelian projection) monopoles are both topological objects that are associated with low-energy phenomena in QCD. While instantons provide a solution to the $U(1)$ problem [1, 2] and an explanation for chiral symmetry breaking [3], they have not yet been able to explain colour confinement [4, 5]. A possible mechanism for the latter is the dual Meissner effect due to condensation of magnetic monopoles that arise in so-called Abelian gauges [6, 7]. Lattice simulations indicate that magnetic monopoles do indeed play an important role in confinement [8–11]. Since lattice simulations also indicate that the transition to a deconfined phase and the restoration of chiral symmetry occur at approximately the same temperature, it would be puzzling if they were generated by completely independent mechanisms. There is indeed evidence from a number of studies both in the continuum and on the lattice that instantons and monopoles are correlated in several Abelian gauges (see, e.g., [12–17]). A connection between the instanton number (Pontryagin index) and magnetic charges has already been considered in [18, 19]. The detailed relation between the total instanton number and the quantum numbers of magnetic monopoles has so far only been established in the Polyakov gauge (or the related modified axial gauge) [20–22].

In the standard model, Taubes has shown how monopole fields can be used to generate topological charge [23]. As pointed out by van Baal in [24], similar arguments may be made in the context of an Abelian projection in pure Yang–Mills theory. This has been demonstrated explicitly for a new finite-temperature instanton (caloron) solution by Kraan and van Baal in [25]. There, it has been shown that the instanton number is carried by a magnetic monopole that makes a full rotation in colour space along its (closed) worldline. It has been noted that the relevant topology for this 'twist' of the monopole is the Hopf fibration. These observations are worked out in greater detail for general configurations in this paper.

Section 2 presents a short review of the definition of general Abelian gauges in terms of an auxiliary Higgs field and of the characterization of the magnetic monopole singularities arising in these gauges. In section 3, the general relation between the instanton number and

the auxiliary Higgs field is established for the Euclidean 'spacetime' S^4. Section 4 provides a generalization of the Hopf invariant of maps from S^3 to S^2 to maps from $S^2 \times S^1$ to S^2. This invariant is used in section 5 to derive the contribution of a single monopole loop to the instanton number. The resulting relation between the instanton number and the generalized Hopf invariants of monopoles is illustrated with the example of a single-instanton solution that is known to lead to a monopole loop in the (differential) maximal Abelian gauge [13]. In section 6, the contribution of topologically non-trivial monopole loops to the instanton number on the spacetime $S^3 \times S^1$ is derived. Section 7 gives a qualitative explanation for the existence of a stronger relation between instantons and monopoles in the Polyakov gauge. The final section contains a discussion of the results.

2. Monopoles in general Abelian gauges

Throughout this work, we consider pure $SU(2)$ Yang–Mills theory. The term 'Abelian gauge' will be used for gauges that are defined by the diagonalization of some field $\phi[A]$ that transforms according to the adjoint representation of the gauge group,

$$\phi(x) \to \Omega(x)\,\phi(x)\,\Omega^+(x) \tag{1}$$

under a gauge transformation $\Omega(x) \in SU(2)$. Due to this property, we will call ϕ an *auxiliary Higgs field*. It is not a fundamental field of the theory but rather a functional of the gauge potential A. The field ϕ can take values in either the gauge group (in our case $SU(2)$) or its algebra ($\mathfrak{su}(2)$). Well known examples are the Polyakov gauge where ϕ is the (time-dependent) Polyakov line,

$$\phi(\boldsymbol{x}, t) = \mathrm{P}\exp\left(\int_t^{t+\beta} \mathrm{d}t'\, A_0(\boldsymbol{x}, t')\right) \in SU(2) \tag{2}$$

(on a spacetime with finite temporal extent β), and the maximal Abelian gauge where $\phi = \phi \cdot \boldsymbol{\sigma} \in \mathfrak{su}(2)$ minimizes the functional

$$R[\phi, A] = \int \mathrm{d}^4x\ \mathrm{tr}\left([\partial_\mu + A_\mu, \phi]^2\right) \tag{3}$$

under the constraint $|\phi| = 1$.

Monopole singularities arise where ϕ does not define a direction in colour space, i.e. where $\phi = 0$ for $\phi \in \mathfrak{su}(2)$ or $\phi = \pm 1$ for $\phi \in SU(2)$. Since these conditions involve three equations, the monopole singularities will generically occupy points in three-dimensional space or one-dimensional submanifolds (worldlines) in four-dimensional spacetime. Around these points, the direction of the auxiliary Higgs field defines a map from a two-dimensional sphere S^2 to another S^2. (In spacetime, one has to consider 2-spheres that link with the monopole worldline.) The winding number of this map provides the charge of the magnetic monopole singularity that appears in the diagonal part of the gauge potential after gauge fixing. It can be expressed as

$$m = \deg[\hat{\phi}] = \frac{1}{8\pi} \int_{S^2} \epsilon_{ijk}\hat{\phi}_i\, \mathrm{d}\hat{\phi}_j \wedge \mathrm{d}\hat{\phi}_k = \frac{-\mathrm{i}}{16\pi} \int_{S^2} \mathrm{tr}\,\hat{\phi}\,\mathrm{d}\hat{\phi} \wedge \mathrm{d}\hat{\phi} \tag{4}$$

where the unit vector $\hat{\phi}$ in the direction of the Higgs field is defined via the relations

$$\phi = \phi \cdot \boldsymbol{\sigma} \qquad \text{and} \qquad \hat{\phi} = \frac{\phi}{|\phi|} \qquad \text{for} \quad \phi \in \mathfrak{su}(2)$$
$$\phi = \cos\beta + \mathrm{i}\hat{\phi} \cdot \boldsymbol{\sigma}\sin\beta \qquad\qquad \text{for} \quad \phi \in SU(2) \tag{5}$$

and $\hat{\phi} = \hat{\phi} \cdot \boldsymbol{\sigma}$ denotes the corresponding $\mathfrak{su}(2)$ matrix.

Using the fact that the gauge-fixing transformation Ω diagonalizes $\hat{\phi}$,

$$\Omega \hat{\phi} \Omega^+ = \sigma_3 \tag{6}$$

m can be expressed in terms of Ω,

$$m = \frac{\mathrm{i}}{4\pi} \int_{S^2} \mathrm{tr}\, \sigma_3 (\mathrm{d}\Omega\, \Omega^+)^2. \tag{7}$$

Since the integrand is a total differential, $(\mathrm{d}\Omega\, \Omega^+)^2 = \mathrm{d}(\mathrm{d}\Omega\, \Omega^+)$, Ω has to be discontinuous at some point x_1 on S^2 if $m \neq 0$. This is the origin of the Dirac string singularity in the Abelian projected gauge potential. Since the Higgs field is continuous on S^2, the discontinuity in Ω has to be Abelian,

$$\Omega(x) \to \mathrm{e}^{-\mathrm{i}\psi(x)\sigma_3} \Omega_0 \qquad \text{for} \quad x \to x_1. \tag{8}$$

The magnetic charge can be expressed as the winding number of the phase ψ along an infinitesimal closed curve $\mathcal{C}$ around x_1 on S^2,

$$m = \frac{1}{2\pi} \int_{\mathcal{C}} \mathrm{d}\psi. \tag{9}$$

Note that although the above discussion does not directly apply to the maximal Abelian gauge since the constraint $|\phi| = 1$ does not permit zeros of ϕ, discontinuities of $\hat{\phi}$ cannot, in general, be avoided also in this gauge and monopole singularities arise after gauge fixing. In this case, of course, the auxiliary Higgs field itself is discontinuous.

3. Instantons in general Abelian gauges

The above discussion shows that all information about the positions and charges of the monopoles is present in the auxiliary Higgs field that defines the Abelian gauge in question. One is prompted to ask whether information about the number of instantons is also included. Since the latter relates to global properties of the gauge field it is useful to consider a specific spacetime geometry. For simplicity, we choose S^4. It can be covered by two charts. We will use one large chart that covers all of S^4 with the exception of one point and as a second chart a small neighbourhood of that point. The excluded point can be chosen such that the direction of the Higgs field is well defined on the small chart. In the overlap, the gauge fields on the two charts are related by a gauge transformation with a transition function $U \in SU(2)$,

$$A^{(1)} = U^+(A^{(2)} + \mathrm{d})U. \tag{10}$$

Since the Higgs field transforms according to the adjoint representation of the gauge group (it belongs to an associated fibre bundle), the Higgs fields on the two charts are related by the same gauge transformation,

$$\phi^{(1)} = U^+ \phi^{(2)} U. \tag{11}$$

We use stereographic projection to parametrize the large chart by $\mathbb{R}^4$. Equation (10) then turns into the statement that $A^{(1)}$ approaches a pure gauge at infinity,

$$A^{(1)}(x) \sim U^+(\hat{x})\, \mathrm{d}U(\hat{x}) \qquad \text{for} \quad |x| \to \infty. \tag{12}$$

We drop the superscript (1) in the following because we no longer need the second chart. The winding number (or degree) of U as a map from S^3 to $SU(2) \cong S^3$ is the total instanton number ν of A,

$$\nu = \deg[U] \equiv \frac{1}{24\pi^2} \int_{S^3} \mathrm{tr}\left[(U^+ \mathrm{d}U)^3\right]. \tag{13}$$

The Higgs field approaches the corresponding gauge transform of a constant (the value of the Higgs field on the excluded point of S^4),

$$\phi(x) \to \phi_\infty(\hat{x}) \equiv U^+(\hat{x})\, \phi_0\, U(\hat{x}) \qquad \text{for} \quad |x| \to \infty. \tag{14}$$

Due to our choice of charts, the direction of ϕ_∞ is well defined. It provides a map from S^3 to S^2. Such maps fall into different homotopy classes and can be characterized by the so-called *Hopf invariant* (see, e.g., [26]). It is usually defined in an indirect way: let ω_2 denote the volume form on S^2 (strictly speaking, the pull-back of it),

$$\omega_2 = \tfrac{1}{2}\epsilon_{ijk}\hat{\phi}_i\, \mathrm{d}\hat{\phi}_j \wedge \mathrm{d}\hat{\phi}_k = -\tfrac{1}{4}\mathrm{i}\, \mathrm{tr}\, \hat{\phi}\, \mathrm{d}\hat{\phi} \wedge \mathrm{d}\hat{\phi}. \tag{15}$$

Since ω_2 is closed and the second cohomology group of S^3 is trivial, ω_2 can be written as a total derivative, $\omega_2 = \mathrm{d}\upsilon$, where υ is a 1-form. The Hopf invariant is defined as

$$\alpha[\hat{\phi}_\infty] \equiv \frac{1}{16\pi^2} \int_{S^3} \upsilon \wedge \mathrm{d}\upsilon \tag{16}$$

and is independent of the choice of υ. Geometrically, the Hopf invariant is given by the linking number of the preimages of two arbitrary points on S^2. The preimages are generically one-dimensional curves and have an orientation induced from the neighbourhood of the two points. The linking number is defined as the number of times one has to cross the two preimages to disentangle them with orientations taken properly into account. It has the algebraic representation

$$l = \frac{1}{4\pi} \int \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \cdot \left(\mathrm{d}\boldsymbol{l} \times \mathrm{d}\boldsymbol{l}'\right) \tag{17}$$

where the line integrals are performed over the two preimages. One can show that l is independent of the choice of the two points on S^2.

The representation (14) can be used to express ω_2 in terms of U,

$$\omega_2 = \mathrm{i}\, \mathrm{tr}[\hat{\phi}_0(\mathrm{d}U\, U^+)^2] \tag{18}$$

which can be easily integrated,

$$\omega_2 = \mathrm{d}\upsilon \qquad \text{with} \quad \upsilon = \mathrm{i}\, \mathrm{tr}[\hat{\phi}_0\, \mathrm{d}U\, U^+]. \tag{19}$$

Without loss of generality we may choose $\hat{\phi}_0 = \sigma_3$ yielding

$$\begin{aligned}
\upsilon \wedge \mathrm{d}\upsilon &= -\, \mathrm{tr}[\sigma_3\, \mathrm{d}U\, U^+] \wedge \mathrm{tr}[\sigma_3\, \mathrm{d}U\, U^+ \wedge \mathrm{d}U\, U^+] \\
&= -(\mathrm{d}U\, U^+)_3 \wedge \tfrac{1}{2}\mathrm{i}\epsilon_{3ij}(\mathrm{d}U\, U^+)_i \wedge (\mathrm{d}U\, U^+)_j \\
&= -\tfrac{1}{6}\mathrm{i}\epsilon_{ijk}(\mathrm{d}U\, U^+)_i \wedge (\mathrm{d}U\, U^+)_j \wedge (\mathrm{d}U\, U^+)_k \\
&= -\tfrac{2}{3}\, \mathrm{tr}\left[(\mathrm{d}U\, U^+)^3\right]
\end{aligned} \tag{20}$$

where the anticommutativity of the wedge product has been exploited. We find that the Hopf invariant is given by the negative of the degree of U,

$$\alpha[\hat{\phi}_\infty] = -\deg[U] = -\nu. \tag{21}$$

The instanton number is therefore identical to the negative of the Hopf invariant of the auxiliary Higgs field at infinity.

How does the latter relate to monopoles? The necessity of points where $\hat{\phi}$ is undefined for non-vanishing instanton number follows immediately: a non-trivial $\hat{\phi}_\infty: S^3 \to S^2$ cannot be deformed into a constant continuously and is therefore not extendible to $\mathbb{R}^4$. The question

of whether these points are monopoles (i.e. have non-zero magnetic charge) and how their charges relate to the instanton number requires a more detailed analysis.

Before this, we investigate how the instanton number decomposes into contributions from the individual monopoles. Consider the generic case of an arbitrary number of closed monopole loops in S^4. Since loops cannot link in four dimensions, it is possible to enclose the individual loops in disjoint 4-volumes V_i that are topologically trivial (have no holes). The Hopf invariant has the nice property of being additive in the sense that $\alpha[\hat{\phi}_\infty]$ can be written as the sum of the Hopf invariants of $\hat{\phi}$ on the boundaries of the volumes V_i,

$$-\nu = \alpha[\hat{\phi}_\infty] = \sum_i \alpha\big[\hat{\phi}\big|_{\partial V_i}\big] \tag{22}$$

since $\hat{\phi}$ is continuous outside of the V_i. Furthermore, since (the adjoint of) the gauge-fixing transformation Ω that diagonalizes ϕ is related to $\hat{\phi}$ in the same way as U to $\hat{\phi}_\infty$,

$$\hat{\phi} = \Omega^+ \sigma_3 \Omega \tag{23}$$

the individual contributions are identical to the respective degrees of Ω,

$$\alpha\big[\hat{\phi}\big|_{\partial V_i}\big] = -\nu\big[\Omega\big|_{\partial V_i}\big]. \tag{24}$$

The right-hand side is non-zero only if Ω is singular in V_i, in which case the degree equals the instanton number of the gauge singularities produced by Ω inside of V_i. We have reduced the problem to the calculation of the Hopf invariant of a single monopole loop in a topologically trivial volume V.

4. Generalized Hopf invariant

In the modified axial gauge, it is possible to express the instanton number in terms of monopole charges that can be calculated from properties of the auxiliary Higgs field in the vicinity of the monopole worldlines [22]. It would be desirable to establish a similar relation in the general case. Accordingly, we embed each monopole loop into a loop of finite thickness and try to assign a Hopf invariant to $\hat{\phi}$ on the surface T of the thick loop. This surface is a higher-dimensional generalization of a tube and has the topology of $S^2 \times S^1$. The coordinate corresponding to the second factor can be interpreted as the proper time $\tau \in [0, 2\pi]$ (in Euclidean space) of the monopole, the first factor as a sphere surrounding the monopole at fixed τ. In the quest for an invariant of $\hat{\phi}|_T$, we seek a characterization of the homotopy classes of maps $\hat{\phi}: S^2 \times S^1 \to S^2$. These have been studied in [27]. Following the ideas developed there, we give a more explicit discussion that is better suited for our purposes. A first characterization is given by the magnetic charge that we introduced in the previous section. It is the winding number of $\hat{\phi}$ in its first argument for fixed τ. By continuity, it has to be independent of τ. However, on a compact manifold the total magnetic charge vanishes. It is therefore not a good candidate for the instanton number.

The most obvious ansatz for a further invariant, a naive generalization of the Hopf invariant (16), is only possible for $m = 0$: the magnetic charge is given by the integral of the pull-back ω_2 of the volume form on S^2 for fixed τ. For $m \neq 0$, it is therefore not possible to write ω_2 as a total differential. In this case, it is actually not possible to define an integer-valued invariant at all, since it turns out that the homotopy classes of maps $S^2 \times S^1 \to S^2$ with a given magnetic charge m form the group $\mathbb{Z}_{2|m|}$ rather than $\mathbb{Z}$ (as can be inferred from the results of [27]).

However, it is possible to generalize the Hopf invariant to a restricted class of functions $S^2 \times S^1 \to S^2$ with magnetic charge $m \neq 0$. It is this invariant that will enable us to establish a relation between instantons and monopoles in section 5. We consider maps $\phi : S^2 \times S^1 \to S^2$

that map a fixed point on S^2 to another fixed point ϕ_0 on the target S^2 for every value of the second argument. For definiteness, we choose the first point to be the south pole. In polar coordinates (ϑ, φ) on S^2, the restriction therefore reads

$$\hat{\phi}(\vartheta = \pi, \varphi, \tau) = \phi_0 \tag{25}$$

with $\phi_0 \in \mathfrak{su}(2)$ and $|\phi_0| = 1$. (Recall that the target S^2 has been introduced as the unit sphere in $\mathfrak{su}(2)$.) Motivated by the relation (21) between the Hopf invariant and the degree of a diagonalizing gauge transformation for maps $S^3 \to S^2$, we diagonalize $\hat{\phi}$,

$$\hat{\phi} = \Omega^+ \sigma_3 \, \Omega \tag{26}$$

with Ω continuous on $(S^2 \setminus \{\vartheta = \pi\}) \times S^1$. For non-zero magnetic charge m, Ω cannot be chosen to be continuous on all of $S^2 \times S^1$. At the south pole, it has an Abelian discontinuity,

$$\Omega(\vartheta, \varphi, \tau) \to e^{-i\psi(\varphi,\tau)\sigma_3} \, \Omega_0 \qquad \text{for} \quad \vartheta \to \pi \tag{27}$$

related to the ambiguity of multiplying Ω by a diagonal matrix from the left in equation (26). Ω_0 is a constant matrix that diagonalizes ϕ_0, i.e. $\phi_0 = \Omega_0^+\sigma_3\Omega_0$.

Unfortunately, the analogue of the degree for maps from $S^2 \times S^1$ to S^2,

$$\nu[\Omega] \equiv \frac{1}{24\pi^2} \int_{\substack{S^2 \times S^1 \\ \vartheta \neq \pi}} \text{tr}(\Omega^+ \, d\Omega)^3 \tag{28}$$

depends on the choice of the diagonalization matrix Ω. Under a change $\Omega \to \omega\Omega$ with $\omega = e^{i\chi(\vartheta,\varphi,\tau)\sigma_3}$, ν is not invariant, because it is not additive for discontinuous Ω,

$$\nu[\omega\Omega] = \nu[\Omega] + \nu[\omega] + \frac{1}{8\pi^2} \lim_{\varepsilon \to 0} \int_{\vartheta = \pi - \varepsilon} \text{tr} \, d\Omega \, \Omega^+ \wedge \omega^+ d\omega. \tag{29}$$

The winding number of the diagonal function ω vanishes, but the surface term gives a contribution (on the boundary $\vartheta = \pi - \varepsilon$, the coordinate system (φ, τ) is right-handed since $(\vartheta, \varphi, \tau)$ is right-handed on $S^2 \times S^1$)

$$\frac{1}{8\pi^2} \lim_{\varepsilon \to 0} \int_{\vartheta = \pi - \varepsilon} \text{tr} \, d\Omega \, \Omega^+ \wedge \omega^+ \, d\omega = \frac{1}{4\pi} \int d\psi(\varphi, \tau) \wedge d\chi(\pi, \varphi, \tau)$$

$$= n_{\tau = \tau_0}[\psi] \, n_{\varphi = \varphi_0}^{\vartheta = \pi}[\chi] - n_{\varphi = \varphi_0}[\psi] \, n_{\tau = \tau_0}^{\vartheta = \pi}[\chi] \tag{30}$$

where we have introduced Abelian winding numbers, e.g.

$$n_{\varphi = \varphi_0}[\psi] \equiv \frac{1}{2\pi} \int_0^{2\pi} d\tau \, \frac{\partial \psi(\varphi_0, \tau)}{\partial \tau}. \tag{31}$$

The other winding numbers are defined analogously. They do not depend on the values φ_0, respectively, τ_0. The winding number of χ for fixed τ vanishes since χ is continuous for all $\vartheta < \pi$ including the north pole. Hence,

$$\nu[\omega\Omega] = \nu[\Omega] + n_{\tau = \tau_0}[\psi] n_{\varphi = \varphi_0}^{\vartheta = \pi}[\chi]. \tag{32}$$

In the case at hand, the winding number of ψ for fixed τ is just the magnetic charge, $n_{\tau = \tau_0}[\psi] = m$ (cf equation (9)). Since the discontinuous phase $\psi(\varphi, \tau)$ in equation (27) changes by $-\chi(\pi, \varphi, \tau)$, it is therefore possible to define an invariant as

$$\alpha_\varphi[\hat{\phi}] \equiv -\nu[\Omega] - m n_{\varphi = \varphi_0}[\psi]. \tag{33}$$

We will refer to this invariant as the *generalized Hopf invariant* on $S^2 \times S^1$. It constitutes the desired topological invariant for maps $S^2 \times S^1 \to S^2$ with magnetic charge m that fulfil

equation (25). It turns out that (33) is the only invariant and the homotopy classes of such maps form the group $\mathbb{Z}$ [27]. The restriction (25) has increased the number of homotopy classes since it restricts the set of possible deformations. If deformations that violate equation (25) are allowed, maps with α differing by multiples of $2m$ can be deformed into each other. A mathematically more appealing definition of α is given in [27]. It coincides with the more explicit definition given here. A definition that is very similar (and equivalent) to equation (33) has also been introduced in [28].

Note that the generalized Hopf invariant depends on the choice of the coordinate φ, as indicated by the subscript on α: consider, for instance, the coordinate system $(\vartheta, \tilde{\varphi}, \tau)$ with $\tilde{\varphi} = \varphi + k\tau$ and integer k that is an admissible parametrization of $S^2 \times S^1$, too. Under this change of coordinates, $\nu[\Omega]$ is not altered, since the volume element occurring in the integral (28) is invariant. The winding number (31), however, changes,

$$n_{\tilde{\varphi}=\varphi_0}[\psi] = \frac{1}{2\pi} \int_0^{2\pi} d\tau \, \frac{d\psi(\varphi_0 - k\tau, \tau)}{d\tau} = n_{\varphi=\varphi_0}[\psi] - kn_{\tau=\tau_0}[\psi] \qquad (34)$$

because the path $\{\varphi = \varphi_0 - k\tau, \tau \in [0, 2\pi]\}$ along which the change of ψ is calculated, is equivalent to the sum of the original path $\{\varphi = \varphi_0, \tau \in [0, 2\pi]\}$ and a path that winds k times around the negative φ-direction for fixed τ. The generalized Hopf invariant therefore changes by m^2k,

$$\alpha_{\tilde{\varphi}}[\hat{\phi}] = \alpha_{\varphi}[\hat{\phi}] + m^2k. \qquad (35)$$

Furthermore, $\alpha_{\varphi}[\hat{\phi}]$ depends on the point in the factor S^2 of the domain (here the south pole) that is used to formulate the constraint (25). One can show that a different choice changes $\alpha_{\varphi}[\hat{\phi}]$ by $2m \deg[\hat{\phi}|_\Delta]$ with $\Delta = \gamma \times S^1$, where γ is a curve between the old and the new point. To apply the above definition, one has to change the coordinate system such that the new point corresponds to $\vartheta = \pi$, of course.

Geometrically, the generalized Hopf invariant is, as for the original Hopf invariant, given by the linking number of the preimages of two points on the target S^2 if we represent $S^2 \times S^1$ as a filled torus $B^2 \times S^1$ in 3-space with the boundary of the disc B^2 identified to one point—the fixed point that is mapped to ϕ_0 in equation (25) (cf figure 1). The ambiguity arising from different coordinates φ is now replaced by the ambiguity of different embeddings in 3-space. In order to obtain the same definition as equation (33), curves with constant φ on the surface of the filled torus must not 'wind around the torus', i.e. be topologically trivial in the complement of the torus. This fixes a possible 'twist' of the torus. Since, for a charge m configuration, each point has m preimages and a twist links every preimage with every other one, it is obvious that it changes the generalized Hopf invariant by m^2. The example below will show that the generalized Hopf invariant measures the twist of the Higgs field. In view of the relation between internal and real space present in a field with non-zero winding number m, it is not surprising that $\alpha_{\varphi}[\hat{\phi}]$ is also sensitive to a twist in real space.

Example. As an example, consider the following auxiliary Higgs field with magnetic winding number m:

$$\phi(\vartheta, \varphi, \tau) = \begin{pmatrix} \sin\vartheta \cos(m\varphi - k\tau) \\ \sin\vartheta \sin(m\varphi - k\tau) \\ \cos\vartheta \end{pmatrix} \cdot \boldsymbol{\sigma}. \qquad (36)$$
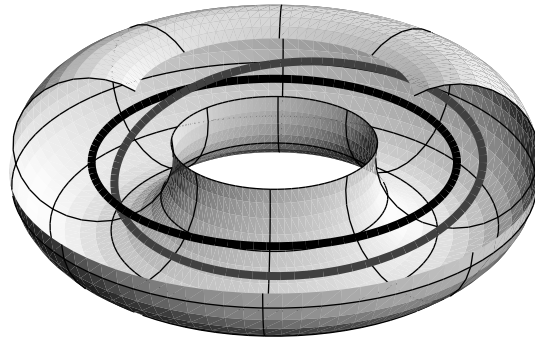
Figure 1. Generalized Hopf invariant as a linking number of preimages in a filled torus in 3-space. The picture shows an example with magnetic charge 1 (one preimage per point) and generalized Hopf invariant 1 (the preimages are linked once).
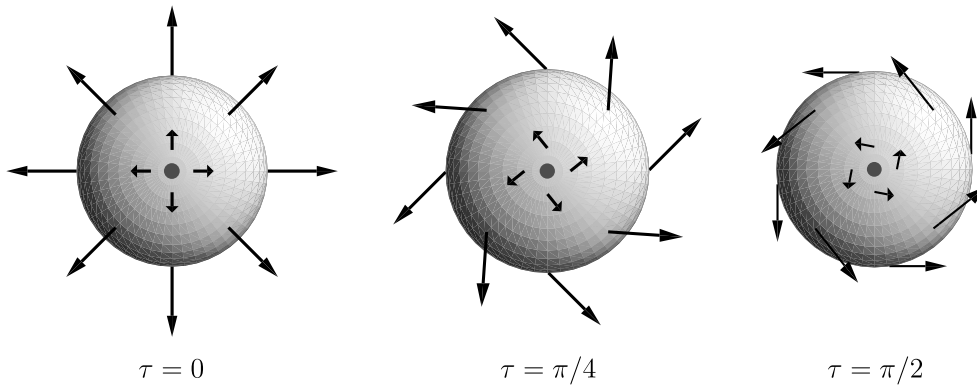

$$\tau = 0 \qquad\qquad \tau = \pi/4 \qquad\qquad \tau = \pi/2$$

Figure 2. Sketch of the Higgs field (36) for $m = k = 1$. The full circle indicates the point that is mapped to ϕ_0 as required by equation (25).

It can be represented as a standard charge m field on S^2 that is 'twisted' around the 3-axis along the worldline of the monopole,

$$\phi(\vartheta, \varphi, \tau) = \omega^+(\tau)\, \phi(\vartheta, \varphi, 0)\, \omega(\tau) \tag{37}$$

$$\omega(\tau) = \mathrm{e}^{-ik\tau\sigma_3/2}. \tag{38}$$

The field ϕ is displayed for some values of τ in figure 2.

Given a diagonalization at $\tau = 0$,

$$\phi(\vartheta, \varphi, 0) = \Omega_1^+(\vartheta, \varphi)\, \sigma_3\, \Omega_1(\vartheta, \varphi) \tag{39}$$

the τ-dependent diagonalizing matrix can be represented as

$$\Omega(\vartheta, \varphi, \tau) = \omega^+(\tau)\, \Omega_1(\vartheta, \varphi)\, \omega(\tau). \tag{40}$$

The factor $\omega^+(\tau)$ is needed to make Ω periodic also for odd k. As argued above, the non-Abelian winding number of Ω is the same as that of Ω_1, because a shift of φ by a multiple of τ does not change it. Since Ω_1 depends on only two parameters, it vanishes, $\nu[\Omega] = 0$. For $\vartheta \to \pi$, one finds

$$\Omega \to \mathrm{e}^{i(k\tau - m\varphi)\sigma_3}\, i\sigma_2 \tag{41}$$

and therefore $n_{\varphi=\varphi_0}[\psi] = k$ and

$$\alpha_\varphi[\hat{\phi}] = mk. \tag{42}$$

We conclude that the generalized Hopf invariant is given by the product of the magnetic charge and the number of times the Higgs field is twisted along the monopole loop. Obviously, the same is true for twists of arbitrary configurations $\phi(\vartheta, \varphi, 0)$. It has been observed that this kind of twist (called 'Taubes winding' in [29]) gives rise to a non-vanishing instanton number [24]. This has been shown explicitly for a finite-temperature instanton with non-trivial holonomy [25]. The following sections investigate this relation in detail for general configurations.

For unit charge monopoles, uniform twists give all possible values of $\alpha_\varphi[\hat{\phi}]$. For higher charges, there are additional cases $0 < |\alpha| < |m|$ that cannot be represented in the simple form (37). They correspond to fields that are twisted only on a part of S^2 that carries one unit (or $m' < m$ units) of magnetic charge.

5. Hopf invariant of a monopole loop

We consider a single closed monopole loop $\mathcal{M}$ where the Higgs field vanishes (or is in the centre for a group-valued field). Following the strategy developed in section 3, we will embed the monopole loop into a topologically trivial 4-volume V. Using equation (22), the contribution of the monopole loop to the instanton number is then given by the Hopf invariant of $\hat{\phi}$ on the surface of V.

As in the previous section, we first embed the monopole loop $\mathcal{M}$ into a loop of finite thickness r,

$$V_{\mathcal{M}} \equiv \left\{ x \in \mathbb{R}^4 \,\middle|\, |x - y| \leqslant r \text{ for some } y \in \mathcal{M} \right\}. \tag{43}$$

r should be so small that $V_{\mathcal{M}}$ does not become topologically non-trivial by self-intersections. Since we intend to apply the definition of the generalized Hopf invariant given above, we choose an isocurve $\mathcal{C}$ of $\hat{\phi}$ on the surface T of $V_{\mathcal{M}}$,

$$\hat{\phi}\big|_{\mathcal{C}} = \hat{\phi}_0 \qquad \text{and} \qquad \mathcal{C} \subset T \equiv \partial V_{\mathcal{M}}. \tag{44}$$

A note on the existence of such a curve: on every section $A \cong S^2$ through T and for every $\vartheta_0 \in (0, \pi)$ there exists a curve γ on which $\hat{\phi} = (\vartheta_0, \varphi)$ and $\int_\gamma d\varphi \neq 0$. Let A^- denote that connected component of $A \setminus \gamma$ where the south pole is taken. On moving A along S^1 through T, A^- changes continuously and cannot disappear because of the non-vanishing integral. The union of all A^- gives an open tube $T_{\vartheta_0}^- \subset T$. On changing ϑ_0, γ, and therefore also $T_{\vartheta_0}^-$, it can be chosen to change continuously. On the intersection $T_\pi^- \equiv \bigcap_{\vartheta_0 \leqslant \pi} T_{\vartheta_0}^-$, we have $\hat{\phi} = s$. Since each section of $T_{\vartheta_0}^-$ along a S^2 in T is simply connected, this is also true for sections of T_π^-. Therefore, T_π^- must contain a curve $\mathcal{C}$ of the required properties.

Now, we close the loop with a two-dimensional sheet D (reminiscent of a Dirac sheet) that has $\mathcal{C}$ as its boundary,

$$\partial D = \mathcal{C} \tag{45}$$

and intersects $V_{\mathcal{M}}$ only there. For $r \to 0$, the condition (44) can be represented in terms of D: it requires that D emerges from $\mathcal{M}$ in a direction where $\hat{\phi} = \hat{\phi}_0$. We complement $V_{\mathcal{M}}$ by a sheet of finite thickness $\varepsilon < r$ around D,

$$V_D \equiv \left\{ x \in \mathbb{R}^4 \,\middle|\, |x - y| \leqslant \varepsilon \text{ for some } y \in D \right\} \cong B^2 \times B^2 \tag{46}$$

to define the topologically trivial volume V,

$$V = V_{\mathcal{M}} \cup V_D. \tag{47}$$

Eventually, we will perform the limit $\varepsilon \to 0$. We decompose the surface of V into parts around the loop and the sheet,

$$\partial V = T_\varepsilon \cup T_{D\varepsilon} \tag{48}$$

$$T_\varepsilon \equiv \overline{T \setminus V_D} \cong B^2 \times S^1 \tag{49}$$

$$T_{D\varepsilon} \equiv \overline{\partial V_D \setminus V_{\mathcal{M}}} \cong S^1 \times B^2. \tag{50}$$

The various manifolds are sketched in figure 3 for the example of a loop in the z–t-plane, $x = y = 0$, $z^2 + t^2 = R^2$, using double polar coordinates in spacetime,

$$x + \mathrm{i}y = u\mathrm{e}^{\mathrm{i}\varphi} \qquad \text{and} \qquad z + \mathrm{i}t = v\mathrm{e}^{\mathrm{i}\tau}. \tag{51}$$

The tube T can be parametrized by the coordinates $\vartheta = \arctan(u/(v - R))$, φ and τ that have the same orientation as in section 4. A double set of polar coordinates (u, φ, v, τ) with $u = 0$ on D and $v = v_0 = \text{constant}$ on $\mathcal{C}$ can be chosen for any $\mathcal{M}$ and D and will be used in the following.
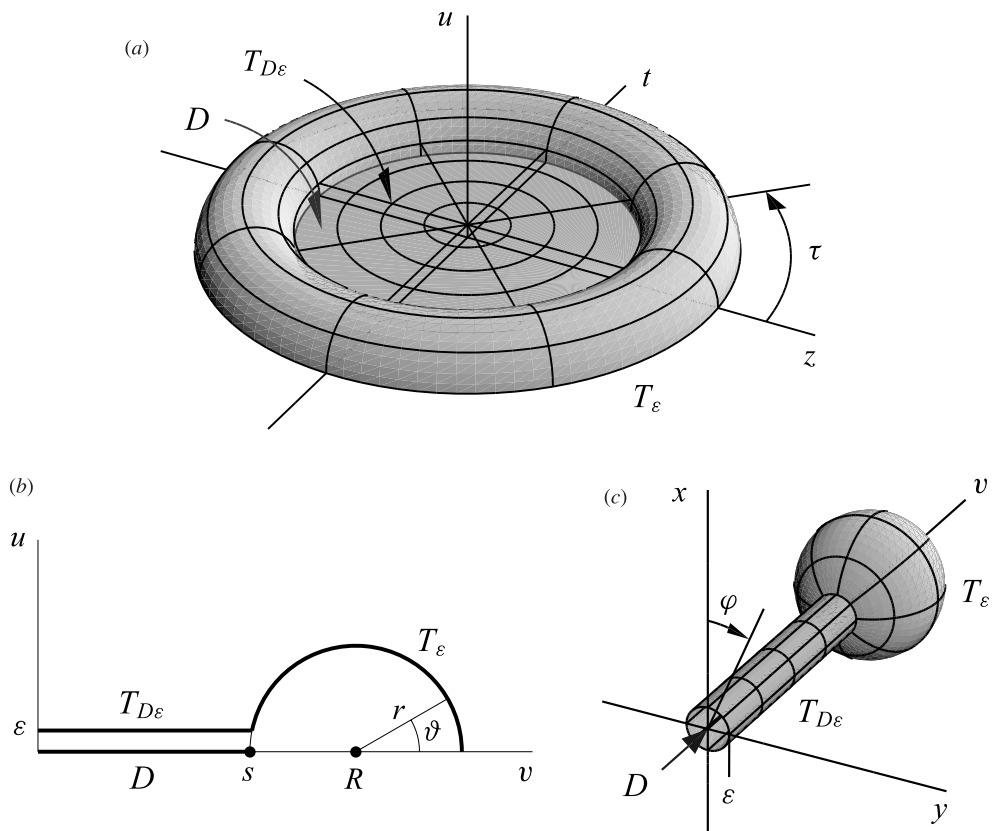
Figure 3. Manifolds used to define V: (*a*) three-dimensional view for fixed φ; (*b*) two-dimensional view for fixed φ and τ; (*c*) three-dimensional view for fixed τ.

The intersection

$$t \equiv T_\varepsilon \cap T_{D\varepsilon} = \partial T_\varepsilon = \partial T_{D\varepsilon} \cong S^1 \times S^1 \tag{52}$$

is parametrized by the coordinates φ and τ.

Since there is no local representation of the Hopf invariant, we cannot calculate separate contributions from T_ε and $T_{D\varepsilon}$ to $\alpha[\hat{\phi}|_{\partial V}]$. Therefore, we diagonalize $\hat{\phi}$ on $\partial V \cong S^3$,

$$\hat{\phi} = \Omega^+ \sigma_3 \Omega \qquad \text{on } \partial V \tag{53}$$

and calculate the contributions to $\nu[\Omega]$, which is by equation (24) equal to the negative of the desired Hopf invariant.

In the limit $\varepsilon \to 0$, the intersection t reduces to the curve $\mathcal{C}$. Condition (44) implies that, in this limit, Ω is constant up to a diagonal factor,

$$\Omega \to e^{-i\psi(\varphi,\tau)\sigma_3} \Omega_0 \quad \text{on } t \qquad \text{for} \quad \varepsilon \to 0. \tag{54}$$

As in section 4, the winding number of ψ for fixed τ gives the magnetic charge of the monopole singularity,

$$n_{\tau=\tau_0}[\psi] = m. \tag{55}$$

The interpretation of the winding number for fixed φ can be found by noting that $T_{D\varepsilon}$ approaches the sheet D in the limit $\varepsilon \to 0$, and therefore

$$\Omega \to e^{i\chi(\varphi,v,\tau)\sigma_3} \tilde{\Omega}(v,\tau) \quad \text{on } T_{D\varepsilon} \qquad \text{for} \quad \varepsilon \to 0 \tag{56}$$

where $\tilde{\Omega}$ is independent of φ and diagonalizes $\hat{\phi}$ on D,

$$\hat{\phi} = \tilde{\Omega}^+ \sigma_3 \tilde{\Omega} \quad \text{on } D. \tag{57}$$

On the boundary $\partial D = \mathcal{C}$, also $\tilde{\Omega}$ is constant up to a diagonal factor,

$$\tilde{\Omega} = e^{-i\tilde{\psi}(\tau)\sigma_3} \Omega_0 \quad \text{on } \mathcal{C}. \tag{58}$$

In the same way that the winding number of ψ is related to the magnetic charge, the winding number of $\tilde{\psi}$ is related to the degree of $\hat{\phi}$ on D (cf equations (4) and (9)),

$$n[\tilde{\psi}] = \deg[\hat{\phi}|_D]. \tag{59}$$

This degree is well defined since the boundary $\mathcal{C}$ of D is mapped to a single point. D is therefore effectively compactified to S^2. It can be interpreted as the flux through D produced by the gauge-fixing transformation. However, since the flux stems from a finite magnetic field, unlike the flux of the monopole singularity, it cannot be distinguished from the flux already present before gauge fixing.

Furthermore, since the two expressions (54) and (56) for Ω on $t = \partial T_{D\varepsilon}$ for $\varepsilon \to 0$ have to coincide, the relation $\psi(\varphi,\tau) = \tilde{\psi}(\tau) - \chi(\varphi,v_0,\tau)$ follows ($v = v_0$ on $\mathcal{C}$). χ is also continuous for $v \to 0$. Therefore, its winding number with respect to τ vanishes and the corresponding winding numbers of ψ and $\tilde{\psi}$ are identical, whence

$$n_{\varphi=\varphi_0}[\psi] = \deg[\hat{\phi}|_D]. \tag{60}$$

The winding number with respect to φ is the negative of that of ψ,

$$n_{\tau=\tau_0}[\chi] = -m. \tag{61}$$

We can now express the contributions to $\nu[\Omega|_{\partial V}]$ in the limit $\varepsilon \to 0$ in terms of $\hat{\phi}$. For T_ε, we insert the winding number of ψ into the definition of the generalized Hopf invariant, equation (33), to obtain

$$\alpha_D[\hat{\phi}|_T] = -\lim_{\varepsilon \to 0} \nu[\Omega|_{T_\varepsilon}] - m \deg[\hat{\phi}|_D]. \tag{62}$$

We have replaced the subscript φ on α by D, because D determines the coordinate φ up to homotopy: φ is that angle on the torus t that can be continuously extended to the whole tube $T_{D\varepsilon}$ around D. Obviously, this is not the case for τ ruling out an admixture of τ to φ.

For the second contribution, we note that, since $T_{D\varepsilon} \cong S^1 \times B^2$ has the same topology as T_ε, we can apply the relation (32) for the non-Abelian winding number of a product to equation (56). The angles τ and φ have exchanged their roles:

$$\lim_{\varepsilon \to 0} \nu[\Omega|_{T_{D\varepsilon}}] = \nu[\tilde{\Omega}] + n_{\varphi=\varphi_0}[\tilde{\psi}] n_{\tau=\tau_0}^{v=v_0}[\chi] = -\deg[\hat{\phi}|_D] m \tag{63}$$

where we have used the fact that $\nu[\tilde{\Omega}]$ vanishes since $\tilde{\Omega}$ depends on only two parameters and have inserted the winding numbers (59) and (61). Putting equations (62) and (63) together, we obtain

$$\lim_{\varepsilon \to 0} \nu[\Omega|_{\partial V}] = \lim_{\varepsilon \to 0} \left(\nu[\Omega|_{T_\varepsilon}] + \nu[\Omega|_{T_{D\varepsilon}}] \right) = -\alpha_D[\hat{\phi}|_T] - 2m \deg[\hat{\phi}|_D]. \tag{64}$$

The Hopf invariant of $\hat{\phi}$ on ∂V is therefore given by

$$\alpha[\hat{\phi}|_{\partial V}] = \alpha_D[\hat{\phi}|_T] + 2m \deg[\hat{\phi}|_D]. \tag{65}$$

This is the desired expression for the contribution of a monopole loop to the instanton number (cf equation (22)). While $\alpha_D[\hat{\phi}|_T]$ depends on the position of the sheet D, it is independent of the values of $\hat{\phi}$ on D, as indicated. The latter enter through the term $\deg[\hat{\phi}|_D]$, though. The instanton number is therefore not given by properties of the auxiliary Higgs field near the monopole singularity only. The instanton number modulo $2m$, however, is

$$\alpha[\hat{\phi}|_{\partial V}] = \alpha_D[\hat{\phi}|_T] \pmod{2m}. \tag{66}$$

One can show that the dependence on the position of D also disappears here, since a different choice of D changes $\alpha_D[\hat{\phi}|_T]$ only by multiples of $2m$: we have already seen in section 4 that a change of the curve $\mathcal{C}$ used for the condition (44) generates such a shift. There, however, a change of the coordinate φ produced a shift by m^2. Here, only a shift by $2m^2$ is possible. The reason is that the embedding of $S^2 \times S^1$ into $\mathbb{R}^4$ given by T fixes the coordinate φ up to multiples of 2τ (up to homotopy). Figure 4, for instance, shows an alternative choice of the sheet D for the loop of figure 3. Consider first a sheet that stays at the position indicated in the first picture for all τ. A curve of constant φ corresponds to a τ-independent point on the circle where the tube meets the sphere. Now consider a sheet that winds once around the sphere, while τ changes by 2π as indicated in the other pictures. Since φ has to be continuous on $T_{D\varepsilon}$, a curve of constant φ has to be homotopically equivalent to a τ-independent point in the $v = 0$-plane. This is indicated by the thick lines on the tubes for a point on the positive x-axis. On the intersection of tube and sphere, the curve of constant φ now winds twice around the circle as τ changes by 2π. This sheet therefore corresponds to a new coordinate $\tilde{\varphi} = \varphi + 2\tau$. Obviously a shift by only τ is not possible.

Consequently, we can assign a unique $\mathbb{Z}_{2|m|}$-valued generalized Hopf invariant to $\hat{\phi}|_{T_i}$,

$$\alpha[\hat{\phi}|_T] \equiv \alpha_D[\hat{\phi}|_T] \bmod 2m \in \mathbb{Z}_{2|m|} \tag{67}$$

$\tau = 0$ $\qquad\qquad \tau = \frac{1}{4}\pi$ $\qquad\qquad \tau = \frac{1}{2}\pi$
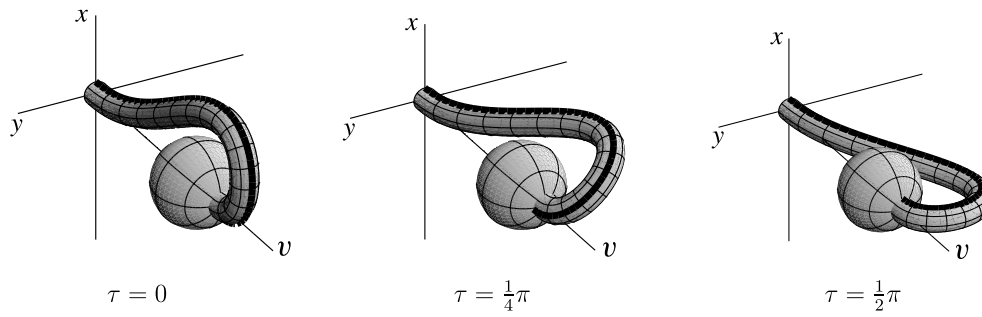
Figure 4. Alternative choice of the sheet D. For details see text.

and write

$$\alpha[\hat{\phi}|_{\partial V}] \bmod 2m = \alpha[\hat{\phi}|_T]. \tag{68}$$

Since the group of homotopy classes of maps from $S^2 \times S^1$ to S^2 with magnetic winding number m is $\mathbb{Z}_{2|m|}$ (cf section 4), this is the maximal information that can be expected.

One could argue that it is possible to get rid of the additional term in equation (65) by choosing a sheet D on which $\hat{\phi}$ is constant, $\hat{\phi}|_D = \hat{\phi}_0$. However, this is not possible in general. If the Hopf invariant $\alpha[\hat{\phi}|_{\partial V}]$ is non-zero, $\hat{\phi}$ takes all possible values on ∂V. This implies that the preimages of all points extend to the exterior of V (and some even to infinity if the total instanton number is non-zero). One therefore has to expect that an isosurface D whose boundary is a monopole loop also leaves V. Such a D cannot be used to identify the contribution of an individual monopole loop to the instanton number in the way described here.

The result (65) can also be understood geometrically: the decomposition $\partial V = T_\varepsilon \cup T_{D\varepsilon}$ corresponds topologically to the decomposition of S^3 into two filled tori, $S^3 = B^2 \times S^1 \cup S^1 \times B^2$ (cf figure 5). The Hopf invariant of $\hat{\phi}$ on ∂V is given by the linking number of the preimages of two points. Each point has $m = \deg[\hat{\phi}|_T]$ preimages in the filled torus corresponding to T_ε and $\deg[\hat{\phi}|_D]$ preimages in the one corresponding to $T_{D\varepsilon}$ if the orientation of the preimages is taken into account. If we furthermore choose the decomposition into the two tori compatible with the coordinate φ around $\mathcal{C}$ in the same way as the embedding of the torus into $\mathbb{R}^3$ in section 4, the linking number of the preimages in T_ε is given by $\alpha_D[\hat{\phi}|_T]$. The preimages in $T_{D\varepsilon}$ do not link since $\hat{\phi}$ becomes φ independent in the limit $\varepsilon \to 0$. Finally, we have to take into account the linking between the preimages in T_ε and $T_{D\varepsilon}$. This gives the remaining term $2m \deg[\hat{\phi}|_D]$ in equation (65).

Monopole loop for the instanton solution. The authors of [13] have found solutions to the differential maximal Abelian gauge condition for the single-instanton solution [30, 31] that correspond to closed monopole loops of various radii. Although the global minimum of the gauge-fixing functional (3) is only reached in the limit of zero radius, it is conjectured that a small perturbation from, for example, a nearby instanton can stabilize a finite radius. To check our result†, we calculate the contributions to equation (65) for the explicit solution that has been given in [13] for the limit in which the radius of the monopole loop is much smaller than the radius of the instanton. In the double polar coordinates of equation (51) in spacetime and

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† In the case of the maximal Abelian gauge, it is also valid for the spacetime $\mathbb{R}^4$, because the finiteness of the gauge-fixing equation (3) guarantees the validity of equation (14).
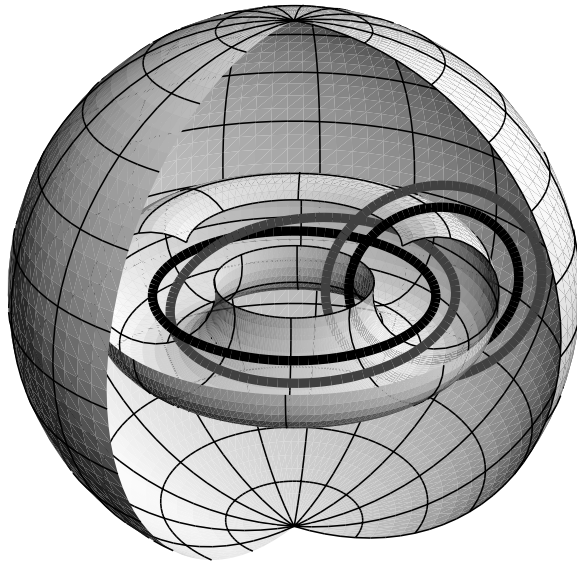
Figure 5. Decomposition $S^3 = B^2 \times S^1 \cup S^1 \times B^2$ and representation of $\alpha[\hat{\phi}|_{\partial V}]$ as a linking number of preimages. S^3 is represented as a 3-ball with its surface identified at a point.
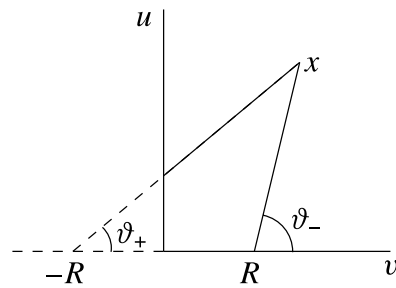

Figure 6. Variables ϑ_+ and ϑ_-.

in spherical polar coordinates in target space, the solution for a regular gauge instanton reads

$$\phi = \begin{pmatrix} \sin\beta \cos(\varphi + \tau) \\ \sin\beta \sin(\varphi + \tau) \\ \cos\beta \end{pmatrix} \cdot \sigma \tag{69}$$

where β is a function of u and v only,

$$\beta(u, v) = \vartheta_+ + \vartheta_- \qquad \text{where} \quad \tan\vartheta_\pm = \frac{u}{v \pm R}. \tag{70}$$

The angles $\vartheta_\pm$ can be chosen as continuous modulo 2π everywhere with the exception of the circle $u = 0$, $v = R$, where the monopole singularity arises (cf figure 6 copied from [13]). A contour plot of $\beta(u, v)$ is shown in figure 7. Since β tends to 0 or π for $u \to 0$ or $v \to 0$, there are no additional singularities due to the angles φ and τ. For $u^2 + v^2 \to \infty$, ϕ tends to the standard Hopf map [32] with τ substituted by $-\tau$ and therefore carries a Hopf invariant of -1. A gauge transformation that diagonalizes ϕ removes the instanton winding number from
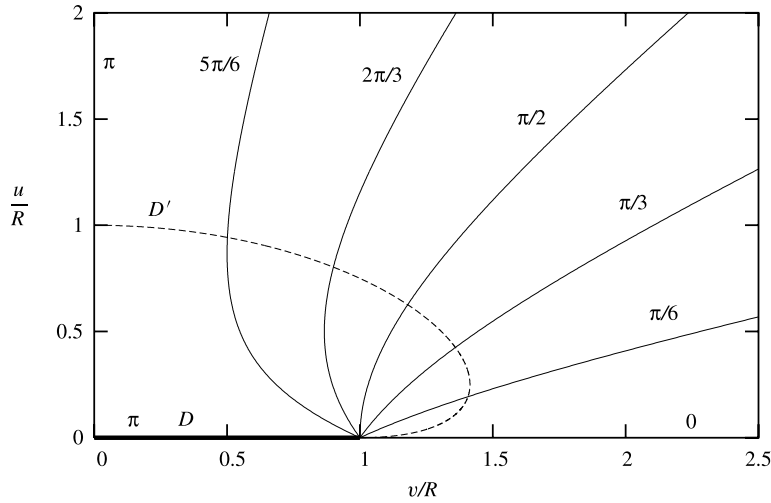
Figure 7. Contour plot of the polar angle parameter $\beta(u, v)$. Two alternative choices (D and D') of the Dirac sheet are included.

the gauge potential at infinity and produces a gauge singularity along the monopole loop (and on a sheet D) that carries the same winding number.

In order to calculate the contributions to equation (65), we note that near the monopole $\vartheta_+ \to 0$ and ϑ_- complements φ to a set of spherical polar coordinates on the sphere around the monopole. Finally, τ measures the position along the monopole loop. A natural choice for the sheet D is $u = 0$, $v \leqslant R$ as in figure 3 where $\vartheta_- = \pi$ and $\phi = -\sigma_3$. The condition (44) is therefore fulfilled for every tube around the monopole loop. Furthermore, the coordinate φ is compatible with the sheet D since it can be defined globally on a tube $u = \varepsilon$, $v \leqslant R$ around the sheet D. Since $\beta \sim \vartheta_-$ near the monopole loop, ϕ is identical to the field (36) from the example in section 4 for $m = k = 1$ and the first term in equation (65) is therefore

$$\alpha_D\big[\phi\big|_T\big] = -1. \tag{71}$$

On the sheet D, ϕ is constant. The corresponding degree therefore vanishes,

$$\deg\big[\hat{\phi}\big|_D\big] = 0 \tag{72}$$

and the second term in (65) does not contribute. We obtain the expected result $\nu = 1$. Note that this is a non-generic case where the argument of the paragraph after equation (68) is circumvented in a special way: while indeed the preimages of all points extend to infinity, that of $\beta = \pi$ splits at the origin into the plane $v = 0$ and the sheet D due to a vanishing Jacobi matrix of ϕ.

To see how the contributions to the instanton number depend on the sheet D chosen, we repeat the calculation for an alternative sheet D' indicated schematically in figure 7. The indicated relation between u and v is complemented by the condition $\varphi = \varphi_0$. The angle ψ is not constrained. By using the right-handed set of coordinates $(\vartheta' = \pi - \vartheta_-, \varphi' = -\varphi, \tau)$ and the formulae in section 4, one finds that in this case

$$\alpha_{D'}\big[\phi\big|_T\big] = +1. \tag{73}$$

The Higgs field on the sheet D' is no longer constant but takes all values on S^2 as can be seen in figure 7,

$$\phi\big|_{D'} = \begin{pmatrix} \sin(\pi - \rho)\cos(\varphi_0 + \tau) \\ \sin(\pi - \rho)\sin(\varphi_0 + \tau) \\ \cos(\pi - \rho) \end{pmatrix} \tag{74}$$

where ρ is a suitable radial coordinate with the range $[0, \pi]$ on D'. Due to the occurrence of $\pi - \rho$, this map has the degree

$$\deg\big[\phi\big|_{D'}\big] = -1. \tag{75}$$

The magnetic charge is still $+1$ because we have not changed the orientation of τ. The Hopf invariant on the surface $\partial V'$ around T and D' is therefore again $\alpha[\phi|_{\partial V'}] = 1 - 2 = -1$. The contributions from the generalized Hopf invariant and the Higgs field on the sheet D, however, have changed.

By 'twisting' the sheet D', i.e. replacing the condition $\varphi = \varphi_0$ by $\varphi = \varphi_0 + n\tau$, one can obtain any odd value $\alpha'_D[\phi|_T] = 2n + 1$ and the appropriate value $\deg[\phi|_D] = -n - 1$ that yield a total $\alpha[\phi|_{\partial V}] = -1$.

6. Topologically non-trivial monopole loops

The procedure of closing the individual loops by sheets cannot be applied to loops that are topologically non-trivial in spacetime. The simplest geometry where this can occur is $S^3 \times S^1$. Topologically non-trivial loops wind around the second factor. For simplicity, we assume all fields to be periodic in the second factor. This can always be accomplished by a gauge transformation. We map S^3 by a stereographic projection to $\mathbb{R}^3$ such that there is no monopole at the point that is mapped to infinity. In this case, the fields tend to a pure gauge at infinity,

$$A(x) \sim U^+(\hat{x}, t)\,\mathrm{d}U(\hat{x}, t)$$
$$\phi(x) \to U^+(\hat{x}, t)\,\phi_0\,U(\hat{x}, t) \equiv \phi_\infty(\hat{x}, t) \qquad \text{for} \quad |x| \to \infty \tag{76}$$

and the instanton number is given by the winding number of the map $U : S^2 \times S^1 \to SU(2)$ which can be expressed as an integral over the same density as for maps $S^3 \to SU(2)$,

$$\nu = \nu[U] \equiv \frac{1}{24\pi^2} \int_{S^2 \times S^1} \mathrm{tr}\big[(U^+\,\mathrm{d}U)^3\big]. \tag{77}$$

Since the total magnetic charge on the compact manifold S^3 necessarily vanishes, $\hat{\phi}_\infty : S^2 \times S^1 \to S^2$ has magnetic winding number zero. As already mentioned in section 4, the set of homotopy classes of such maps is $\mathbb{Z}$ and is parametrized by a Hopf invariant defined in an analogous way as for maps $S^3 \to S^2$. Consequently, the relation between the winding number of U and the Hopf invariant of $\hat{\phi}_\infty$ also remains the same,

$$\nu[U] = -\alpha[\hat{\phi}_\infty]. \tag{78}$$

However, the procedure advocated in section 3 is not directly applicable here because it is not possible to embed topologically non-trivial monopole loops into topologically trivial volumes. If we embed a single topologically non-trivial loop into a topologically non-trivial volume V, the auxiliary Higgs field has a non-zero magnetic winding number on ∂V. It is therefore not possible to assign a unique Hopf invariant to it. The best we can do in order to decompose the Hopf invariant, is to group the monopole loops into neutral sets and embed

each set into a volume that is topologically as simple as possible, i.e. equivalent to $B^3 \times S^1$. If this is done, the Hopf invariant of $\hat{\phi}_\infty$ again splits into contributions from the boundaries of the volumes V_i,

$$-\nu = \alpha[\hat{\phi}_\infty] = \sum_i \alpha[\hat{\phi}|_{\partial V_i}] \tag{79}$$

where topologically trivial loops are treated as before.

To complete the calculation of the instanton number, we only have to consider a single set of N topologically non-trivial monopole loops $\mathcal{M}_i$ with magnetic charges m_i and total charge zero,

$$\sum_{i=1}^{N} m_i = 0. \tag{80}$$

In order to construct the volume $V \cong B^3 \times S^1$ around these, we first embed the individual loops $\mathcal{M}_i$ into thick loops $V_{\mathcal{M}_i}$ with boundaries $T_i \cong S^2 \times S^1$. Then we connect the 'tubes' T_i by $N - 1$ sheets D^β that do not intersect with each other and intersect with the tubes on curves $\mathcal{C}_i^\beta$ where $\hat{\phi}$ is constant (cf figure 8),

$$\begin{aligned} D^\beta \cap D^\gamma &= \emptyset \qquad \text{for} \quad \beta \neq \gamma \\ D^\beta \cap T_i &= \mathcal{C}_i^\beta \subset \partial D^\beta \end{aligned} \tag{81}$$

$$\hat{\phi}|_{\mathcal{C}_i^\beta} = \phi_i^\beta = \text{constant}. \tag{82}$$

We assume that two of the tubes (T_1 and T_N) intersect only with one and the others with two sheets. This means that monopoles and sheets form an open chain. The tubes and sheets will be numbered consecutively.
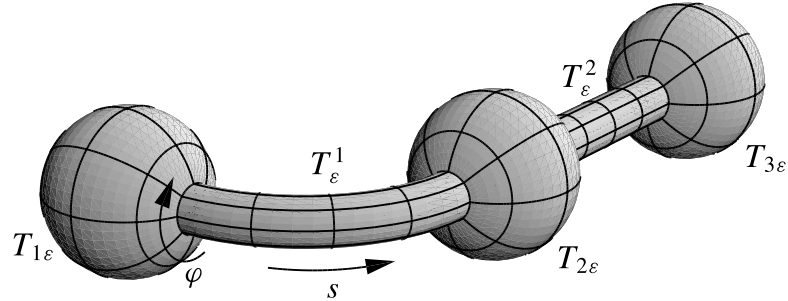

Figure 8. Three-dimensional section of spacetime for fixed τ with tubes around monopoles and sheets.

As for the case of topologically trivial monopole loops, we introduce thick sheets $V_{D^\beta} \cong B^2 \times I \times S^1$ and decompose the surface around the union V of all thick loops and sheets,

$$V \equiv \bigcup_i V_{\mathcal{M}_i} \cup \bigcup_\beta V_{D^\beta} \tag{83}$$

into parts around loops and sheets,

$$\partial V = \bigcup_i T_{i\varepsilon} \cup \bigcup_\beta T_\varepsilon^\beta \tag{84}$$

$$T_{i\varepsilon} \equiv \overline{T_i \setminus \bigcup_\beta V_{D^\beta}} \tag{85}$$

$$T_\varepsilon^\beta \equiv \overline{\partial V_{D^\beta} \setminus \bigcup_i V_{\mathcal{M}_i}}. \tag{86}$$

The topology of these manifolds is as follows:

$$\partial V \cong S^2 \times S^1 \tag{87}$$

$$T_{i\varepsilon} \cong \begin{cases} B^2 \times S^1 & \text{for} \quad i = 1, N \\ S^1 \times I \times S^1 & \text{for} \quad i = 2, \ldots, N-1 \end{cases} \tag{88}$$

$$T_\varepsilon^\beta \cong S^1 \times I \times S^1. \tag{89}$$

The intersections still have the topology of tori,

$$t_i^\beta \equiv T_{i\varepsilon} \cap T_\varepsilon^\beta \cong S^1 \times S^1. \tag{90}$$

We assume ε to be so small that the thick sheets V_{D^β} do not intersect. t_i^β will be parametrized by two angles φ and τ, where τ runs along the monopole loops and can be defined globally on ∂V, while φ measures the angle around the sheets. It can be defined globally on ∂V with the exception of one point on both $T_{1\varepsilon}$ and $T_{N\varepsilon}$. We complement φ and τ with a third coordinate s such that s and φ are spherical polar coordinates on the factor S^2 of ∂V and $s = s_i^\beta$ is constant along the curves C_i^β. Thus, s takes the role of ϑ on T_ε and v on $T_{D\varepsilon}$ in the previous section.

On $\partial V \cong S^2 \times S^1$, we can diagonalize $\hat{\phi}$ continuously, since the magnetic winding number of $\hat{\phi}$ on ∂V vanishes,

$$\hat{\phi} = \Omega^+ \sigma_3 \Omega \quad \text{on } \partial V. \tag{91}$$

In the limit $\varepsilon \to 0$, the intersections t_i^β reduce to the curves C_i^β where Ω has to be constant up to a diagonal factor,

$$\Omega \to e^{-i\psi_i^\beta(\varphi,\tau)\sigma_3} \Omega_0 \quad \text{on } t_i^\beta \qquad \text{for} \quad \varepsilon \to 0. \tag{92}$$

The winding numbers of ψ_i^β are again related to the magnetic winding numbers of $\hat{\phi}$: on the one hand,

$$n_{\tau=\tau_0}[\psi_i^i] - n_{\tau=\tau_0}[\psi_i^{i-1}] = \deg[\hat{\phi}|_{T_i}] = m_i \tag{93}$$

where we have set $\psi_1^0 = \psi_N^N = 0$. On the other hand, T_ε^β approaches D^β as $\varepsilon \to 0$, whence

$$\Omega \to e^{i\chi(\varphi,s,\tau)\sigma_3} \tilde{\Omega}(s,\tau) \quad \text{on } T_\varepsilon^\beta \qquad \text{for} \quad \varepsilon \to 0 \tag{94}$$

where $\tilde{\Omega}$ diagonalizes $\hat{\phi}$ on D^β,

$$\hat{\phi} = \tilde{\Omega}^+ \sigma_3 \tilde{\Omega} \quad \text{on } D^\beta \tag{95}$$

and is constant up to a diagonal factor on the boundary $\partial D = C$,

$$\tilde{\Omega} = e^{-i\tilde{\psi}_i^\beta(\tau)\sigma_3} \Omega_0 \quad \text{on } C_i^\beta. \tag{96}$$

Since $\hat{\phi}$ maps the boundaries C_β^β and $C_{\beta+1}^\beta$ of D^β to the fixed points ϕ_β^β and $\phi_{\beta+1}^\beta$, $\hat{\phi}|_D^\beta$ can be interpreted as a function from S^2 to S^2 and the degree $\deg[\hat{\phi}|_{D^\beta}]$ is well defined. It is also related to the winding numbers of the $\tilde{\psi}_i^\beta$,

$$n[\tilde{\psi}_\beta^\beta] - n[\tilde{\psi}_{\beta+1}^\beta] = \deg[\hat{\phi}|_{D^\beta}]. \tag{97}$$

Equations (92) and (94) imply $\psi_i^\beta(\varphi, \tau) = \tilde{\psi}(\tau) - \chi_i^\beta(\varphi, \tau)$ with $\chi_i^\beta(\varphi, \tau) \equiv \chi(\varphi, s_i^\beta, \tau)$. Since χ interpolates between C_β^β and $C_{\beta+1}^\beta$, its winding numbers on both curves have to be

equal. The winding numbers of ψ with respect to φ are therefore identical at both ends of the sheet,

$$n_{\tau=\tau_0}[\psi_\beta^\beta] = n_{\tau=\tau_0}[\psi_{\beta+1}^\beta] \equiv m^\beta. \tag{98}$$

They can be interpreted as the Abelian magnetic flux carried *along* the string from monopole to monopole as opposed to the flux $\deg[\hat{\phi}|_{D^\beta}]$ that flows perpendicularly *through* the sheet.

Furthermore,

$$n_{\varphi=\varphi_0}[\psi_{\beta+1}^\beta] - n_{\varphi=\varphi_0}[\psi_\beta^\beta] = \deg[\hat{\phi}|_{D^\beta}]. \tag{99}$$

In the limit $\varepsilon \to 0$, we can now relate the non-Abelian winding numbers of Ω on $T_{i\varepsilon}$ and T_ε^β to the Abelian winding numbers and generalized Hopf invariants. For $T_{i\varepsilon}$, we have to express the generalized Hopf invariant in terms of a diagonalizing function Ω that is now discontinuous along *two* curves. Considerations very similar to those in section 4 can be used to verify that the correct generalization of equation (33) is

$$\alpha_\varphi[\hat{\phi}|_{T_i}] = -\nu[\Omega|_{T_i}] - n_{\tau=\tau_0}[\psi_i^i]\,n_{\varphi=\varphi_0}[\psi_i^i] + n_{\tau=\tau_0}[\psi_i^{i-1}]\,n_{\varphi=\varphi_0}[\psi_i^{i-1}]. \tag{100}$$

This expression coincides with the definition (33) applied to a diagonalization of $\hat{\phi}|_{T_i}$ that is discontinuous along either $\mathcal{C}_i^i$ or $\mathcal{C}_i^{i-1}$.

For T_ε^β, we apply the relation (32) to equation (56). In addition to the exchange of φ and τ, we have to take the contributions from two boundaries into account,

$$\lim_{\varepsilon \to 0} \nu[\Omega|_{T_\varepsilon^\beta}] = n[\tilde{\psi}_\beta^\beta]n_{\tau=\tau_0}[\chi_\beta^\beta] - n[\tilde{\psi}_{\beta+1}^\beta]n_{\tau=\tau_0}[\chi_{\beta+1}^\beta]. \tag{101}$$

The total winding number of Ω can now be expressed as

$$\lim_{\varepsilon \to 0} \nu[\Omega|_{\partial V}] = \lim_{\varepsilon \to 0}\left(\sum_i \nu[\Omega|_{T_{i\varepsilon}}] + \sum_\beta \nu[\Omega|_{T_\varepsilon^\beta}] \right)$$

$$= -\sum_i \alpha_\varphi[\hat{\phi}|_{T_i}] - 2\sum_\beta \left(n_{\tau=\tau_0}[\psi_\beta^\beta]\,n_{\varphi=\varphi_0}[\psi_\beta^\beta] - n_{\tau=\tau_0}[\psi_{\beta+1}^\beta]\,n_{\varphi=\varphi_0}[\psi_{\beta+1}^\beta] \right). \tag{102}$$

By equation (78), $-\nu[\Omega|_{\partial V}]$ is equal to the Hopf invariant of $\hat{\phi}|_{\partial V}$. Inserting the expressions (98) for $n_{\tau=\tau_0}[\psi_i^\beta]$ and (99) for $n_{\varphi=\varphi_0}[\psi_i^\beta]$, we therefore obtain the final result for the contribution to equation (79),

$$\alpha[\hat{\phi}|_{\partial V}] = \sum_i \alpha_\varphi[\hat{\phi}|_{T_i}] - 2\sum_\beta m^\beta \deg[\hat{\phi}|_{D^\beta}]. \tag{103}$$

m^β can be calculated from m_i by use of the relation (93) which can be expressed as

$$m_i = m^i - m^{i-1} \tag{104}$$

with $m^0 = m^N = 0$:

$$m^\beta = \sum_{i=1}^\beta m_i. \tag{105}$$

Therefore, equation (103) contains information about $\hat{\phi}$ only and is independent of the choice of Ω. Note, that although the generalized Hopf invariants of $\hat{\phi}$ on the tubes around the individual monopole loops depend on the choice of the coordinate φ, their sum is determined by the sheets D^β that relate φ on the various tubes (cf figure 8). As in the case of topologically trivial monopole loops, the instanton number modulo $2m$, where m is the largest common divisor of m_i, is determined by the auxiliary Higgs field on the tubes T_i around the monopoles only, and is independent of the sheets D^β chosen.

7. Polyakov gauge

We consider the Polyakov gauge (or the related modified axial gauge) on the spacetime $S^3 \times S^1$ with periodic boundary conditions in time. In this set-up, a stronger relation between the instanton number and monopoles holds [20–22],

$$\nu = - \sum_{\substack{i \\ \phi(x_i)=-1}} m_i \tag{106}$$

where the sum is taken over all monopole singularities where the Polyakov line is -1. In contrast to the general case, the position of the Dirac strings does not enter and every monopole contributes only $\pm m_i$ (or 0) to the instanton number. Two monopoles with charges ± 1 and Polyakov line ± 1, for instance, give $\nu = \pm 1$ depending on the combination of signs. Our above result, on the other hand, suggests that each of the two monopoles can have an arbitrary 'twist', and therefore every integral value of the instanton number should be possible. The Polyakov line must determine the relative twist of the monopoles in some way. In this section, we try to shed some light on this connection.

A special property of the Polyakov gauge is that the Polyakov line (cf equation (2)) at a *single* time, e.g. $t = 0$, already contains some information on its time *dependence*: first, the eigenvalues of the Polyakov line are time independent, since its time evolution is given by

$$\phi(\boldsymbol{x}, t) = U^+(\boldsymbol{x}, t)\,\phi(\boldsymbol{x}, 0)\,U(\boldsymbol{x}, t) \tag{107}$$

where $U(\boldsymbol{x}, t) \in SU(2)$ is the parallel transporter from $(\boldsymbol{x}, 0)$ to $(\boldsymbol{x}, t)$ along a straight line. This relation implies that the monopoles are static. Second, the temporal boundary conditions of U are given in terms of $\phi(\boldsymbol{x}, 0)$,

$$\begin{aligned} U(\boldsymbol{x}, 0) &= 1 \\ U(\boldsymbol{x}, \pi) &= \phi(\boldsymbol{x}, 0) \end{aligned} \tag{108}$$

where we have chosen the temporal extension of spacetime to be π. The boundary conditions on U, of course, restrict the possible time dependence of ϕ. It turns out that this restriction determines the instanton number.

As before, the charts on S^3 (or the stereographic projection) are chosen such that there is no monopole at spatial infinity,

$$\phi(\boldsymbol{x}, 0) \to \phi_\infty(\hat{\boldsymbol{x}}) \neq \pm 1 \qquad \text{for} \quad |\boldsymbol{x}| \to \infty. \tag{109}$$

Since the transition function at $|\boldsymbol{x}| \to \infty$ and $t = 0$ maps S^2 to $SU(2)$ and is therefore homotopically trivial, $\phi_\infty(\hat{\boldsymbol{x}})$ can always be made constant by a gauge transformation. This will be assumed in the following. Inside the chart, $U: \mathbb{R}^3 \times I \to SU(2)$ is a continuous function. On the boundary of its domain $\mathbb{R}^3 \times I$, U has the following values:

$$U(\boldsymbol{x}, t) = \begin{cases} 1 & \text{for} \quad t = 0 \\ \phi(\boldsymbol{x}, 0) & \text{for} \quad t = \pi \\ U_\infty(\hat{\boldsymbol{x}}, t) & \text{for} \quad |\boldsymbol{x}| \to \infty. \end{cases} \tag{110}$$

Continuity of U implies

$$\begin{aligned} U_\infty(\hat{\boldsymbol{x}}, 0) &= 1 \\ U_\infty(\hat{\boldsymbol{x}}, \pi) &= \phi_\infty. \end{aligned} \tag{111}$$

U_∞ can therefore be interpreted as a function from S^3 to $SU(2)$, and since U is continuous, its winding number must be the opposite of the winding number of $\phi(\boldsymbol{x}, 0)$,

$$n[U_\infty] = -n[\phi(\cdot, 0)]. \tag{112}$$

Since U_∞ is not periodic, it cannot be used to formulate a boundary condition for the gauge field by itself. We therefore introduce

$$\tilde{U}_\infty(\hat{\boldsymbol{x}}, t) \equiv e^{-i\alpha_\infty \cdot \sigma t/\pi} U_\infty(\hat{\boldsymbol{x}}, t) \qquad \text{with} \quad e^{i\alpha_\infty \cdot \sigma} = \phi_\infty. \tag{113}$$

This function is periodic, and since $e^{-i\alpha_\infty \cdot \sigma} \phi_\infty e^{i\alpha_\infty \cdot \sigma} = \phi_\infty$, we still have

$$\phi(\boldsymbol{x}, t) \to \tilde{U}_\infty^+(\hat{\boldsymbol{x}}, t)\, \phi_\infty\, \tilde{U}_\infty(\hat{\boldsymbol{x}}, t) \qquad \text{for} \quad |\boldsymbol{x}| \to \infty. \tag{114}$$

Therefore, the instanton number is given by the winding number of $\tilde{U}_\infty^+$,

$$\nu = -n[\tilde{U}_\infty] = -n[U_\infty] = n[\phi(\cdot, 0)]. \tag{115}$$

We conclude that the Polyakov line at a single time contains enough information about its time dependence to determine the instanton number. The above considerations also apply to a volume enclosing a neutral set of monopoles. The Polyakov line at a single time therefore really determines the 'relative twist' (the contribution (103) to ν) of such a set.

If we drop the requirement (108), the most general boundary condition for U compatible with periodicity of ϕ is

$$U(\boldsymbol{x}, \pi) = e^{i\beta(\boldsymbol{x})\hat{\alpha}(\boldsymbol{x}) \cdot \sigma} \qquad \text{where} \quad \phi(\boldsymbol{x}, 0) = e^{i\alpha(\boldsymbol{x}) \cdot \sigma}. \tag{116}$$

For U to be continuous, we must have

$$\beta(\boldsymbol{x}) = k\pi \qquad \text{with} \quad k \in \mathbb{Z} \qquad \text{if} \quad \phi(\boldsymbol{x}, 0) = \pm 1. \tag{117}$$

The relation between $U(\boldsymbol{x}, \pi)$ and the instanton number is, of course, still valid.

For the choice $\beta(\boldsymbol{x}) = k\alpha(\boldsymbol{x})$, i.e. $U(\boldsymbol{x}, \pi) = \big(\phi(\boldsymbol{x}, 0)\big)^k$, for instance, the winding number of $U(\cdot, \pi)$ and therefore the instanton number is multiplied by k,

$$\nu = n[U(\cdot, \pi)] = kn[\phi(\cdot, 0)]. \tag{118}$$

With other choices of β, all values of ν can be generated as long as monopoles are present. In general Abelian gauges, the Higgs field at a single time does not therefore determine the instanton number, even if its eigenvalues are time independent.

8. Discussion

In this work, the instanton number has been expressed in terms of the auxiliary Higgs field defining a general Abelian gauge. On the spacetime S^4, the instanton number can be written as a sum over contributions associated with individual monopole loops,

$$\nu = -\sum_i \alpha\big[\hat{\phi}\big|_{\partial V_i}\big] \tag{119}$$

where V_i is a topologically trivial volume containing the monopole loop in question. The contribution of a monopole of magnetic charge m to the instanton number modulo $2m$ is given in terms of the Higgs field near the monopole singularity, only,

$$\alpha\big[\hat{\phi}\big|_{\partial V}\big] \bmod 2m = \alpha\big[\hat{\phi}\big|_T\big] \in \mathbb{Z}_{2|m|} \tag{120}$$

where T is a small tube around the monopole loop and $\alpha[\hat{\phi}|_T]$ measures the 'twist' ('Taubes winding') of the Higgs field on that tube. For uniform twist, it is given by the product of the magnetic charge and the number of times the configuration is twisted as one passes along the loop. For the generic case of unit charge monopoles, $\alpha[\hat{\phi}|_T]$ determines the instanton number modulo 2, i.e. whether it is odd or even.

The full instanton number can also be expressed in terms of the Higgs field, however, not exclusively in terms of the values near monopole singularities,

$$\alpha[\hat{\phi}|_{\partial V}] = \alpha_D[\hat{\phi}|_T] + 2m \deg[\hat{\phi}|_D] \tag{121}$$

where D denotes a (Dirac) sheet closing the monopole loop. The generalized Hopf invariant $\alpha_D[\hat{\phi}|_T]$ depends on the position of the sheet but on values of $\hat{\phi}$ only on T; it has the same interpretation as $\alpha[\hat{\phi}|_T]$. The values of $\hat{\phi}$ away from the monopole loop enter through the degree $\deg[\hat{\phi}|_D]$ of $\hat{\phi}$ on the sheet D. The total contribution to the instanton number, equation (121), is independent of the choice of D.

For unit charge monopoles, the $\mathbb{Z}_2$ contribution (120) can be related to the centre symmetry: an odd twist (i.e. one contributing to ν mod 2) can be generated by applying a gauge transformation that changes by a factor of -1 as one passes once along the monopole loop. Such a discontinuity does not affect the gauge potential that transforms according to the adjoint representation of the gauge group. For non-trivial loops on $S^3 \times S^1$, this can be interpreted as a centre symmetry transformation that is applied to only one of the monopoles but not to the others. This is only possible if a singularity is produced between the monopoles, or if the field between the monopoles is altered in a way that does not correspond to a gauge transformation. Of course, such a change is necessary to alter the instanton number. For a topologically trivial monopole loop, the gauge transformation has to be discontinuous along a two-dimensional surface that links with the monopole loop. It produces a 'centre-vortex' singularity on the sheet. If the singularity is avoided by altering the fields, a 'thick centre vortex' is generated (or removed). In a recent work [33] it has been shown that in a continuum version of the maximal centre gauge the instanton number can be related to self-intersections of centre-vortices. The total number of self-intersections is only non-zero if a (connected) vortex contains regions with different orientations. Since the orientation of a vortex (as defined in [33]) can only change at the worldline of a magnetic monopole, it should be possible to express the number of self-intersections as the linking number of vortices with monopoles. Our findings indicate that a similar relation may be valid in other centre gauges, like, for example, the Laplacian centre gauge proposed in [34].

Acknowledgments

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